Green-Kubo formula for weakly coupled system with dynamical noise.

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GREEN-KUBO FORMULA FOR WEAKLY COUPLED SYSTEM WITH DYNAMICAL NOISE.

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Abstract. We consider an infinite system of cells coupled into a chain by a smooth nearest neighbor potential εV. The uncoupled system (cells) evolve according to Hamiltonian dynamics perturbed stochastically with an energy conserving noise of strenght ς. We study the Green-Kubo (GK) formula $\kappa(\varepsilon, \varsigma)$ for the heat conductivity of this system which exists and is finite for $\varsigma > 0$, by formally expanding $\kappa(\varepsilon, \varsigma)$ in a power series in ε, $\kappa(\varepsilon, \varsigma) = \sum_{n \geq 2} \varepsilon^n \kappa_n(\varsigma)$. We show that $\kappa_2(\varsigma)$ is the same as the conductivity obtained in the weak coupling (van Hove) limit where time is rescaled as $\varepsilon^{-2}t$.

$\kappa_2(\varsigma)$ is conjectured to approach as $\varsigma \to 0$ a value proportional to that obtained for the weak coupling limit of the purely Hamiltonian chain.We also show that the $\kappa_2(\varsigma)$ from the GK formula, is the same as the one obtained from the flux of an open system in contact with Langevin reservoirs. Finally we show that the limit $\varsigma \to 0$ of $\kappa_2(\varsigma)$ is finite for the pinned anharmonic oscillators due to phase mixing caused by the non-resonating frequencies of the neighboring cells. This limit is bounded for coupled rotors and vanishes for harmonic chain with random pinning.

1. Introduction

Energy transport in nonequilibrium macroscopic systems is described phenomenologically by Fourier's law. This relates the energy flux J, at the position r in the system, to the temperature gradient at r, via $J = -\kappa \nabla T$. The computation of the thermal conductivity κ, which depends on the temperature and the constitution of the system, from

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the underlying microscopic dynamics is one of the central mathematical problems in nonequilibrium statistical mechanics (see [5][21][9] and references therein).

The Green-Kubo (GK) formula gives a linear response expression for the thermal conductivity. It is defined as the asymptotic space-time variance for the energy currents in an infinite system in equilibrium at temperature $T = \beta^{-1}$, evolving according to the appropriate dynamics. For purely Hamiltonian (or quantum) dynamics, there is no proof of convergence of the GK formula (and consequently no proof of Fourier law). One way to overcome this problem is to add a dash of randomness (noise) to the dynamics [3]. In the present work we explore the resulting GK formula and start an investigation of what happens when the strength of the noise, ς, goes to zero.

Our basic setup is a chain of coupled systems. Each uncoupled system (to which we will refer as a *cell*) evolves according to Hamiltonian dynamics (like a billiard, a geodesic flow on a manifold of negative curvature, or an anharmonic oscillator...) perturbed by a dynamical energy preserving noise, with intensity ς. We will consider cases where the only conserved quantity for the dynamics with $\varsigma > 0$, is the energy. The cells are coupled by a smooth nearest neighbor potential εV. We assume that the resulting infinite volume Gibbs measure has a convergent expansion in ε for small ε. We are interested in the behaviour of the resulting GK formula for $\kappa(\varepsilon, \varsigma)$ given explicitly by equation (2.4) below, for small ς and ε keeping the temperature β^{-1} and other parameters fixed.

We start in Section 2 by noting that for $\varsigma > 0$, the GK formula is well defined and has a finite upper bound [3]. We do not however have a strictly positive lower bound on $\kappa(\varepsilon, \varsigma)$ except in some special cases [3]. We believe however that $\kappa(\varepsilon, \varsigma) > 0$ whenever $\varepsilon > 0, \varsigma > 0$, i.e. there are no (stable) heat insulators. The fact that $\kappa(\varepsilon, \varsigma) \geq 0$ follows from the definition of the GK formula. The situation is different when we let $\varsigma \to 0$. In that case we have examples where $\kappa(\varepsilon, \varsigma) \to 0$ (disordered harmonic chains [2]), and where $\kappa(\varepsilon, \varsigma) \to \infty$ (periodic harmonic systems).

To make progress in elucidating the properties of $\kappa(\varepsilon, \varsigma)$, when $\varsigma \to 0$, we carry out in Section 3 a purely formal expansion of $\kappa(\varepsilon, \varsigma)$ in powers of ε: $\kappa(\varepsilon, \varsigma) = \sum_{n \geq 2} \kappa_n(\varsigma)\varepsilon^n$. This is formal because space-time correlations entering in the GK formula are non-local function and depends themselves on ε.

We then investigate in Section 4 the structure of the term $\kappa_2(\varsigma)$, which we believe, but do not prove, coincides with the $\lim_{\varepsilon \to 0} \kappa(\varepsilon, \varsigma)/\varepsilon^2$.

We show that $\kappa_2(\varsigma)$ is finite and strictly positive for $\varsigma > 0$ by proving that it is equal to the conductivity obtained from a weak coupling limit in which there is a rescaling of time as $\varepsilon^{-2}t$ (cf. [22, 23]). We argue further that the $\lim_{\varsigma\to 0}\kappa_2(\varsigma)$ exists and is closely related to the weak coupling macroscopic conductivity obtained for the purely Hamiltonian dynamics $\varsigma = 0$ from the beginning. The latter is computed for a geodesic flow on a surface of negative curvature, and is strictly positive [12]. A proof, in the latter case, would require the extension to random perturbations for the theory developed for deterministic perturbations in [7][6]. This should be possible by arguing as in the discrete time case [18].

Nevertheless the identification of $\kappa_2(\varsigma)$ with the weak coupling limit conductivity (suggested by H. Spohn [25]) gives some hope that the higher order terms, can also be shown to be well defined and studied in the limit $\varsigma \to 0$. This could then lead (if nature and mathematics are kind) to a proof of the convergence and positivity of the GK formula for a Hamiltonian system.

We next show in Section 5 that we obtain the same $\kappa_2(\varsigma)$ for the thermal conductivity of an open system: N coupled cells in which cell 1 and cell N are in contact with Langevin reservoirs at different temperatures, when we let $N \to \infty$ and the two reservoir temperatures approach to β^{-1}.

Section 6 is devoted to a detailed study of $\kappa_2(\varsigma)$ for 3 examples: the isolated cell hamiltonian is 1) a pinned anharmonic oscillator, 2) a rotor; 3) the system at $\varsigma = 0$ is a random (positively) pinned harmonic chain. In all cases we can prove that, generically, $\limsup_{\varsigma\to 0}\kappa_2(\varsigma) < +\infty$, as contrasted with the regular harmonic chain when $\kappa_2(\varsigma) \to \infty$ when $\varsigma \to 0$ [1][10]. In case 1 and 2 we have no lower bound for this limit, but we believe that it will be strictly positive. In case 3 we prove that the limit is 0, as in the harmonic chain for $\varsigma \to 0$, with random pinning springs. Phase mixing, due to lack of resonances between frequencies of different cells at different energies, is the relevant ingredient for the finiteness of $\kappa_2(\varsigma)$ when $\varsigma \to 0$.

2. Green-Kubo formula (random dynamics)

We define first the dynamics of a single uncoupled cell. This will be given by a Hamiltonian dynamics generated by

$$\mathcal{H} = p^2/2 + W(q)$$

where the position q has values in some d-dimensional manifold, $q \in M$, and the momentum $p \in \mathbb{R}^d$. We generally assume that $W \geq 0$, and its minimum value is 0. In the case of the dynamics of a billiard, $W = 0$

and $M \subset \mathbb{R}^2$ is the corresponding compact set of allowed position with reflecting condition on the boundary. Another chaotic example is given by M a manifold with negative curvature and $W = 0$ (cf [12]). We will also consider cases where $d = 1$, that are completely integrable.

The Hamiltonian flow in a cell is perturbed by a noise that typically acts on the velocity, conserving the kinetic and the internal energy of each cell (as in [3][22]). Noises that exchanges energy betweens different cells will not be considered here. Consequently the energy current will be due entirely to the deterministic interaction between the cells.

The time evolved $\{q(t), p(t)\}$ is given by a Markov process on the state space $\Omega = M \times \mathbb{R}^d$, generated by

$$L = A + \varsigma S$$

where A is the Liouville operator associated to the Hamiltonian flow and S is the generator of the stochastic perturbation. We assume that S acts only on the momentum p and is such that $S|p|^2 = 0$. For $d \geq 2$, we just take for S the Laplacian on the sphere $|p|^2 =$ constant. In dimension 1, we take an S that generates at random exponential times a flip on the sign of the velocity:

$$Sf(q,p) = f(q,-p) - f(q,p).$$

In all cases p is an eigenfunction of S for some negative eigenvalue: $Sp = -\lambda p$.

Consider now the dynamics on $\Omega^{\mathbb{Z}}$ constituted by infinitely many processes $\{q_x(t), p_x(t)\}_{x\in\mathbb{Z}}$ as above, but coupled by a smooth nearest neighbor potential εV. The dynamics is then generated by[1]

$$L_\varepsilon = \sum_{x\in\mathbb{Z}} \left[\varsigma S^x + A_0^x + \varepsilon \nabla V(q_x - q_{x-1})(\partial_{p_{x-1}} - \partial_{p_x})\right] = L_0 + \varepsilon G \quad (2.1)$$

where $L_0 = A_0 + \varsigma S$, $A_0 = \sum_x A_0^x$, $S = \sum_x S^x$.

The energy of each cell, which is the sum of the internal energy and of the interaction energy, is defined by

$$e_x^\varepsilon = e_x + \frac{\varepsilon}{2}\left(V(q_{x+1} - q_x) + V(q_x - q_{x-1})\right). \quad (2.2)$$

To simplify notation we write e_x for e_x^0, the energy of the isolated system x.

This dynamics conserves the total energy. The corresponding energy currents $\varepsilon j_{x,x+1}$, defined by the local conservation law

$$L_\varepsilon e_x^\varepsilon = \varepsilon\left(j_{x-1,x} - j_{x,x+1}\right)$$

[1]Note that, in general, we should write $V(q_x, q_{x-1})$ as q might not belong to a vector space. We avoid it to simplify notation, see [12] for details.

are antisymmetric functions of the p's such that

$$j_{x,x+1} = -\frac{1}{2}(p_x + p_{x+1}) \cdot \nabla V(q_{x+1} - q_x). \tag{2.3}$$

Let us denote by $\mu_{\beta,\varepsilon} = \langle \cdot \rangle_{\beta,\varepsilon}$ the canonical Gibbs measure at temperature $\beta^{-1} > 0$ defined by the Dobrushin-Lanford-Ruelle equations, which of course depends on the interaction εV. we shall assume in all the cases considered that $\mu_{\beta,\varepsilon}$ is analytical in ε for sufficiently small ε (when applied to local functions). In particular we assume that the potentials V and W are such that the Gibbs state is unique and has spatial exponential decay of correlations. Technical assumptions on V and W for this to hold can for example be found in [4]. Also we assume that the infinite dynamics is well defined for a set of initial conditions which has probability measure one with respect to $\mu_{\beta,\varepsilon}$ (we refer the interested reader to [14, 15, 16] and references therein).

The argument of Section 5 in [3] applies here, and gives the convergence of the thermal conductivity defined by the Green-Kubo formula

$$\kappa(\varepsilon, \varsigma) = \varepsilon^2 \int_0^\infty \sum_{x \in \mathbb{Z}} \mathbb{E}_{\beta,\varepsilon}\left(j_{x,x+1}(t) j_{0,1}(0)\right) dt. \tag{2.4}$$

Here $\mathbb{E}_{\beta,\varepsilon}$ indicates the expectation of the infinite dynamics in equilibrium at temperature β^{-1}. The convergence of the integral in (2.4) is in fact defined as

$$\lim_{\nu \to 0} \ll j_{0,1}, (\nu - L_\varepsilon)^{-1} j_{0,1} \gg_{\beta,\varepsilon} \tag{2.5}$$

for $\nu > 0$, where $\ll \cdot, \cdot \gg_{\beta,\varepsilon}$ is the inner product

$$\ll f, g \gg_{\beta,\varepsilon} = \sum_{x \in \mathbb{Z}} [< \tau_x f, g >_{\beta,\varepsilon} - < f >_{\beta,\varepsilon} < g >_{\beta,\varepsilon}].$$

By the same argument as in [3], we have the bound

$$\sup_{\nu > 0} \ll j_{0,1}, (\nu - L_\varepsilon)^{-1} j_{0,1} \gg_{\beta,\varepsilon} \le \frac{C}{\varsigma} \tag{2.6}$$

where C is independent of ε. The convergence in (2.5) and the bound (2.6) are based on the fact that $j_{0,1}$ being a linear function on the momentum p, is an eigenfunction of the generator of the noise S. It then fluctuates fast in time making the integral in the GK formula convergent.

3. Formal expansion of $\kappa(\varepsilon, \varsigma)$

It follows from (2.6) that $\kappa(\varepsilon, \varsigma)$ is of order ε^2, i.e.

$$\limsup_{\varepsilon \to 0} \varepsilon^{-2} \kappa(\varepsilon, \varsigma) = \hat{\kappa}_2(\varsigma) < +\infty, \tag{3.1}$$

We conjecture that the limit exists and it is given by $\kappa_2(\varsigma)$, the lowest term in the formal expansion of $\kappa(\varepsilon, \varsigma)$ in powers of ε:

$$\kappa(\varepsilon, \varsigma) = \sum_{n=2}^{\infty} \varepsilon^n \kappa_n(\varsigma). \tag{3.2}$$

It turns out that it is convenient for calculating the terms in this expansion to choose $\nu = \varepsilon^2 \lambda$ in (2.6), for a $\lambda > 0$, and solve the resolvent equation

$$(\lambda \varepsilon^2 - L_\varepsilon) u_{\lambda,\varepsilon} = \varepsilon j_{0,1} \tag{3.3}$$

for the unknown function $u_{\lambda,\varepsilon}$. The reason for considering $\lambda > 0$ is to have well defined solutions also for the infinite system. The factor ε^2 is the natural scaling in view of the subsequent computations.

Next, define $\mathbf{e} = \{e_x, x \in \mathbb{Z}\}$. For any function $f \in L^1(\mu_{\beta,0})$,

$$(\Pi f)(\mathbf{e}) = \mu_{\beta,0}(f|\mathbf{e}), \quad Q = \mathbb{1} - \Pi.$$

We will look for solutions of (3.3) of the form

$$u_{\lambda,\varepsilon} = \sum_{n \geq 0} (v_{\lambda,n} + w_{\lambda,n}) \varepsilon^n, \tag{3.4}$$

where $\Pi v_{\lambda,n} = Q w_{\lambda,n} = 0$. Observe that $L_0 w_{\lambda,n} = 0$ and that $\Pi G \Pi = 0$, where G is defined by (2.1). Accordingly

$$\begin{aligned} & v_{\lambda,0} = 0 \\ & -L_0 v_{\lambda,1} - G w_{\lambda,0} = j_{0,1} \\ & \lambda w_{\lambda,n-2} + \lambda v_{\lambda,n-2} - L_0 v_{\lambda,n} - G v_{\lambda,n-1} - G w_{\lambda,n-1} = 0, \qquad n \geq 2. \end{aligned} \tag{3.5}$$

It follows from the last equation, since $\Pi L_0 v_{\lambda,n} = 0$, that $\Pi G v_{\lambda,n-1} = \lambda w_{\lambda,n-2}$. In addition,

$$\begin{aligned} G v_{\lambda,1} &= G(-L_0)^{-1}(G w_{\lambda,0} + j_{0,1}) \\ G v_{\lambda,n} &= G(-L_0)^{-1}\left[-\lambda v_{\lambda,n-2} + G w_{\lambda,n-1} + Q G v_{\lambda,n-1}\right]. \end{aligned}$$

It is then natural to consider the operator

$$\mathcal{L} = \Pi G (-L_0)^{-1} G \Pi. \tag{3.6}$$

Using $\mathcal{L}$, we can write the equations for $v_{\lambda,n}$ and $w_{\lambda,n}$ in the form

$$\begin{aligned} & w_{\lambda,0} = (\lambda - \mathcal{L})^{-1}\left[\Pi G(-L_0)^{-1} j_{0,1}\right] \\ & v_{\lambda,1} = (-L_0)^{-1}[j_{0,1} + G w_{\lambda,0}] \\ & w_{\lambda,n} = (\lambda - \mathcal{L})^{-1} \Pi G (-L_0)^{-1}\left[-\lambda v_{\lambda,n-1} + Q G v_{\lambda,n}\right], \qquad n \geq 1 \\ & v_{\lambda,n+1} = (-L_0)^{-1}\left[-\lambda v_{\lambda,n-1} + G w_{\lambda,n} + Q G v_{\lambda,n}\right], \qquad n \geq 1. \end{aligned} \tag{3.7}$$

Of course, the above expressions are at present only formal. To begin with the term $(-L_0)^{-1}$ is a priori ill defined since the space $\mathcal{K}_0$ composed of functions of the internal energies $\{e_x\,;\,x \in \mathbb{Z}\}$ alone is in the kernel of L_0. However, in (3.7), the operator $(-L_0)^{-1}$ is always applied to a function f that satisfies $\Pi f = 0$, i.e. to a function which is orthogonal to $\mathcal{K}_0$. Therefore, if we assume that $\mathcal{K}_0$ coincides with the kernel of L_0, which should be a consequence of the ergodic properties of the internal dynamics, including the noise $\varsigma > 0$, then the function $g := (-L_0)^{-1} f$, where f is orthogonal to $\mathcal{K}_0$, is well defined. We in fact take g to be the unique solution such that $\Pi g = 0$. We will show in proposition 3.1 below that the operator $\mathcal{L}$ is a generator of a Markov process so that $(\lambda - \mathcal{L})^{-1}$ is well defined.

3.1. **The operator $\mathcal{L}$.**

Let us denote by $\rho_\beta(d\mathbf{e})$ the distribution of the internal energies $\mathbf{e} = \{e_x\,;\,x \in \mathbb{Z}\}$ under the Gibbs measure $\mu_{\beta,0}$. It can be written in the form

$$d\rho_\beta(\mathbf{e}) = \prod_{x\in\mathbb{Z}} Z_\beta^{-1} \exp(-\beta e_x - U(e_x)) de_x$$

for a suitable function U. We denote the formal sum $\sum_x U(e_x)$ by $\mathcal{U} := \mathcal{U}(\mathbf{e})$. We denote also, for a given value of the internal energy $\tilde{e}_x$ in the cell x, by $\nu^x_{\tilde{e}_x}$ the microcanonical probability measure in the cell x. i.e. the uniform probability measure on the manifold

$$\Sigma_{\tilde{e}_x} := \{(q_x, p_x) \in \Omega\,;\, e_x(q_x,p_x) \,=\, \tilde{e}_x\}.$$

Proposition 3.1. *The operator $\mathcal{L}$, when restricted to the range of Π, is given by*

$$\mathcal{L} = \sum_x e^{\mathcal{U}} (\partial_{e_{x+1}} - \partial_{e_x}) \left[e^{-\mathcal{U}} \gamma^2(e_x, e_{x+1}) \right] (\partial_{e_{x+1}} - \partial_{e_x}), \tag{3.8}$$

where

$$\gamma^2(e_0, e_1) = \int_0^\infty dt \int_{\Sigma_{e_0}\times\Sigma_{e_1}} j_{0,1}\,(e^{tL_0} j_{0,1})\, d\nu_{e_0} d\nu_{e_1} \tag{3.9}$$

and e^{tL_0} denotes the semigroup of the uncoupled dynamics generated by L_0. In addition, setting $\alpha(e_x, e_{x+1}) = \Pi G(-L_0)^{-1} j_{x,x+1}$, we have

$$\alpha(e_x, e_{x+1}) = e^{\mathcal{U}(\mathbf{e})} (\partial_{e_{x+1}} - \partial_{e_x}) \left[e^{-\mathcal{U}(\mathbf{e})} \gamma(e_x, e_{x+1})^2 \right]. \tag{3.10}$$

Proof. Let us start by noting that, for each local function f depending only on the energies,

$$\begin{aligned} Gf &= -\sum_x \nabla V(q_x - q_{x-1}) \cdot (p_x \partial_{e_x} - p_{x-1}\partial_{e_{x-1}})f \\ &= \sum_x j_{x-1,x}(\partial_{e_x} - \partial_{e_{x-1}})f - \frac{1}{2}\sum_x [L_0 V(q_x - q_{x-1})](\partial_{e_x} + \partial_{e_{x-1}})f \\ &= \sum_x j_{x-1,x}(\partial_{e_x} - \partial_{e_{x-1}})f - L_0\left(\frac{1}{2}\sum_x V(q_x - q_{x-1})(\partial_{e_x} + \partial_{e_{x-1}})f\right). \end{aligned} \tag{3.11}$$

Note that the above expression is linear in p. Next, note that the adjoint of G in $\mathbb{L}^2(\mu_{\beta,0})$ is given by

$$G^* = -G - \beta\sum_x \nabla V(q_x - q_{x-1})\cdot(p_x - p_{x-1}) = -G - \beta L_0 V = -G + \beta L_0^* V. \tag{3.12}$$

Thus, if f, g are smooth local functions of the energies only then, remembering that the measure is symmetric in p,

$$\begin{aligned} \langle g\mathcal{L}f\rangle_{\beta,0} &= \langle G^* g \cdot (-L_0)^{-1} Gf\rangle_{\beta,0} = -\langle Gg \cdot (-L_0)^{-1}Gf\rangle_{\beta,0} - \beta\langle Vg \cdot Gf\rangle_{\beta,0} \\ &= -\sum_{x,y}\langle j_{y-1,y}(\partial_{e_y} - \partial_{e_{y-1}})g \cdot (-L_0)^{-1} j_{x-1,x}(\partial_{e_x} - \partial_{e_{x-1}})f\rangle_{\beta,0}. \end{aligned}$$

Next, note that

$$\langle j_{y,y-1}(-L_0)^{-1} j_{x,x-1} | \mathbf{e}\rangle_{\beta,0} = \delta_{xy}\gamma^2(e_{x-1}, e_x),$$

and consequently

$$\langle g\mathcal{L}f\rangle_{\beta,0} = -\sum_x \langle \gamma^2(e_{x-1}, e_x)(\partial_{e_x} g - \partial_{e_{x-1}} g)(\partial_{e_x} f - \partial_{e_{x-1}} f)\rangle_{\beta,0}.$$

Thus, for each local smooth function g of the energies only,

$$\begin{aligned} \langle gG(-L_0)^{-1} j_{0,1}\rangle_{\beta,0} &= -\langle Gg \cdot (-L_0)^{-1} j_{0,1}\rangle_{\beta,0} - \beta\langle gVj_{0,1}\rangle \\ &= -\sum_y \langle j_{y-1,y}(\partial_{e_y} - \partial_{e_{y-1}})g \cdot (-L_0)^{-1} j_{0,1}\rangle_{\beta,0} \\ &= -\langle(\partial_{e_1} - \partial_{e_0})g \cdot \gamma(e_0, e_1)^2\rangle_{\beta,0} \\ &= \left\langle g \cdot e^{\mathcal{U}(\mathbf{e})}(\partial_{e_1} - \partial_{e_0})\left[e^{-\mathcal{U}(\mathbf{e})}\gamma^2(e_0, e_1)\right]\right\rangle_{\beta,0} \end{aligned}$$

from which the Lemma follows. □

We conclude this section by noting that the operator $\mathcal{L}$ is the generator of a Ginzburg-Landau dynamics which is reversible with respect to ρ_β, for any $\beta > 0$ [22, 12, 23]. It is conservative in the energy $\sum_x e_x$ and the microscopic current corresponding to this conservation law is given by $\alpha(e_x, e_{x+1})$. The corresponding finite dynamics appears in [22, 12] as the weak coupling limit of a finite number N (fixed) of cells weakly coupled by a potential ϵV in the limit $\epsilon \to 0$ when time t is rescaled as $t\varepsilon^{-2}$. Moreover, the hydrodynamic limit of the Ginzburg-Landau dynamics is then given (in the diffusive scale tN^2, $N \to +\infty$), by a heat equation with diffusion coefficient which coincides with κ_2 as given by (4.12) given below ([26],[23]).

4. The lowest order term $\kappa_2(\varsigma)$

To simplify notation, we denote $\gamma(e_x, e_{x+1})$ (resp. $\alpha(e_x, e_{x+1})$) by $\gamma_{x,x+1}$ (resp. $\alpha_{x,x+1}$). Define the operator $D_{x,x+1} = \gamma_{x,x+1}(\partial_{e_{x+1}} - \partial_{e_x})$, then the adjoint, with respect to ρ_β, is given by

$$D^*_{x,x+1} = -e^{\mathcal{U}}(\partial_{e_{x+1}} - \partial_{e_x})e^{-\mathcal{U}}\gamma_{x,x+1}$$

and consequently $\mathcal{L} = \sum_x D^*_{x,x+1}D_{x,x+1}$. First note that for any $\lambda > 0$, we have by (3.7)

$$\lambda w_\lambda - \mathcal{L}w_\lambda = \alpha_{0,1} = D^*_{0,1}\gamma_{0,1}. \tag{4.1}$$

This resolvent equation, which involves only functions of the energies, has a well defined solution $w_\lambda \in \mathbb{L}^2(\rho_\beta)$ for $\lambda > 0$.

For any function f of the energies, denote by $\Gamma_f = \sum_x \tau_x f$ (intended as a formal sum). Observe moreover that by the definition of ρ_β, since f depends only on the energies, we have $\langle f\rangle_{\beta,0} = \rho_\beta(f)$. In the following we will denote by $\langle\cdot\rangle_\beta$ the integration with respect to ρ_β. Then we have the relations

$$\begin{aligned} \ll f, -\mathcal{L}g \gg_\beta &= \left\langle \gamma_{0,1}^2\left[(\partial_{e_1} - \partial_{e_0})\Gamma_f\right]\left[(\partial_{e_1} - \partial_{e_0})\Gamma_g\right]\right\rangle_\beta \\ &= \left\langle (D_{0,1}\Gamma_f)(D_{0,1}\Gamma_g)\right\rangle_\beta, \\ \ll f, \alpha \gg_\beta &= -\left\langle \gamma_{0,1}D_{0,1}\Gamma_f\right\rangle_\beta. \end{aligned} \tag{4.2}$$

So from (4.1) we have

$$\lambda \ll w_\lambda, w_\lambda \gg_\beta + \left\langle (D_{0,1}\Gamma_{w_\lambda})^2\right\rangle_\beta = -\left\langle \gamma_{0,1}D_{0,1}\Gamma_{w_\lambda}\right\rangle_\beta \tag{4.3}$$

thus by the Schwarz inequality

$$\lambda \ll w_\lambda, w_\lambda \gg_\beta + \left\langle (D_{0,1}\Gamma_{w_\lambda})^2\right\rangle_\beta \le \left\langle \gamma_{0,1}^2\right\rangle^{1/2}\left\langle (D_{0,1}\Gamma_{w_\lambda})^2\right\rangle_\beta^{1/2}$$

and this gives the bounds

$$\lambda \ll w_\lambda, w_\lambda \gg_{\beta,0} \le \left\langle \gamma_{0,1}^2\right\rangle_\beta, \qquad \left\langle (D_{0,1}\Gamma_{w_\lambda})^2\right\rangle_\beta \le \left\langle \gamma_{0,1}^2\right\rangle_\beta.$$

The standard Kipnis-Varadhan argument ([19], [20] chapter 1) then gives

$$\lim_{\lambda\to 0} \lambda \ll w_\lambda, w_\lambda \gg_\beta = 0.$$

It also follows form the same argument ([19], [20] chapter 1) that $D_{0,1}\Gamma_{w_\lambda}$ converges strongly in $L^2(\rho_\beta)$ to a limit that we denote with η and satisfies the relation

$$\left\langle \eta^2 \right\rangle_\beta = - \left\langle \gamma_{0,1}\eta \right\rangle_\beta .$$

We now return to the formal expansion of $\kappa(\varepsilon,\varsigma)$ in powers of ε given by (3.2). By (2.5), (3.4) and the expansion of the canonical Gibbs measure $\mu_{\beta,\varepsilon}$ in ε, we have, at least formally, that

$$\begin{aligned} \kappa(\varepsilon,\varsigma) &\equiv \lim_{\lambda\to 0} \ll u_{\lambda,\varepsilon}, \varepsilon j_{0,1} \gg_{\beta,\varepsilon} = \sum_{n\geq 1} \varepsilon^{n+1} \lim_{\lambda\to 0} \ll j_{0,1}, v_{\lambda,n} \gg_{\beta,\varepsilon} \\ &= \varepsilon^2 \lim_{\lambda\to 0} \ll j_{0,1}, v_{\lambda,1} \gg_{\beta,\varepsilon} \; + o(\varepsilon^2). \end{aligned} \tag{4.4}$$

We now compute

$$\begin{aligned} \ll j_{0,1}, v_{\lambda,1} \gg_{\beta,\varepsilon} &= \ll j_{0,1}, (-L_0)^{-1} j_{0,1} \gg_{\beta,\varepsilon} + \ll j_{0,1}, (-L_0)^{-1} G w_\lambda \gg_{\beta,\varepsilon} \\ &= \ll j_{0,1}, (-L_0)^{-1} j_{0,1} \gg_{\beta,\varepsilon} \\ &\quad + \sum_y \ll j_{0,1}, (-L_0)^{-1} j_{y-1,y} (\partial_{e_y} - \partial_{e_{y-1}}) w_\lambda \gg_{\beta,\varepsilon} \\ &\quad + \sum_y \ll j_{0,1}, V(q_{e_y} - q_{e_{y-1}}) (\partial_{e_y} + \partial_{e_{y-1}}) w_\lambda \gg_{\beta,\varepsilon} \end{aligned} \tag{4.5}$$

where we have used formula (3.11) and we have assumed that $(\partial_{e_y} + \partial_{e_{y-1}})w_\lambda$ and $(\partial_{e_y} - \partial_{e_{y-1}})w_\lambda$ exists and are bounded in L^2 of $\langle\cdot\rangle_{\beta,\varepsilon}$. In fact we only have existence of $(\partial_{e_y} - \partial_{e_{y-1}})w_\lambda$. By a smoothing argument one can handle $(\partial_{e_y} + \partial_{e_{y-1}})w_\lambda$. Then last term in (4.5) is zero by the symmetry in p of the Gibbs measure. Let us compute the first term. By the symmetry in p again we have

$$\ll j_{0,1}, (-L_0)^{-1} j_{0,1} \gg_{\beta,\varepsilon} = \sum_{|x|\leq 1} \langle j_{x,x+1}, (-L_0)^{-1} j_{0,1} \rangle_{\beta,\varepsilon}. \tag{4.6}$$

To compute it is convenient to use the following little Lemma.

Lemma 4.1. *For each integrable function g of the energies $\{e_x\}$ we have*

$$\langle j_{-1,0}, (-L_0)^{-1} j_{0,1} g \rangle_{\beta,\varepsilon} = \mathcal{O}(\varepsilon). \tag{4.7}$$

and

$$\langle j_{0,1}, (-L_0)^{-1} j_{0,1} g \rangle_{\beta,\varepsilon} = \langle \gamma_{0,1}^2 g \rangle_{\beta,\varepsilon} + \mathcal{O}(\varepsilon). \tag{4.8}$$

Proof. Note that the adjoint of L_0 with respect to $\langle\cdot\rangle_{\beta,\varepsilon}$ is given by

$$L_0' f = L_0^* f - \varepsilon\beta \sum_x (p_{x+1} - p_x)\nabla V(q_{x+1} - q_x) f,$$

where L_0^* is the adjoint with respect to $\langle\cdot\rangle_{\beta,0}$. Then, observing that

$$\langle p_{-1} \cdot \nabla V(q_0 - q_{-1}), (-L_0)^{-1} j_{0,1} g\rangle_{\beta,\varepsilon} = 0$$

we have

$$\begin{aligned}
\langle j_{-1,0}, &(-L_0)^{-1} j_{0,1} g\rangle_{\beta,\varepsilon} = \frac{1}{2}\langle (p_0 - p_{-1}) \cdot \nabla V(q_0 - q_{-1}), (-L_0)^{-1} j_{0,1} g\rangle_{\beta,\varepsilon} \\
&= \frac{1}{2}\langle L_0^* V(q_0 - q_{-1}), (-L_0)^{-1} j_{0,1} g\rangle_{\beta,\varepsilon} \\
&= \frac{1}{2}\langle L_0' V(q_0 - q_{-1}), (-L_0)^{-1} j_{0,1} g\rangle_{\beta,\varepsilon} \\
&+ \frac{1}{2}\varepsilon\beta \sum_x \langle (p_{x+1} - p_x)\nabla V(q_{x+1} - q_x) V(q_0 - q_{-1}), (-L_0)^{-1} j_{0,1} g\rangle_{\beta,\varepsilon} \\
&= \mathcal{O}(\varepsilon).
\end{aligned}$$

Proof of (4.8) follows a similar line. $\square$

Applying the above Lemma with $g \equiv 1$ to the terms $x = -1$ and $x = 1$ in (4.6) it follows

$$\ll j_{0,1}, (-L_0)^{-1} j_{0,1} \gg_{\beta,\varepsilon} = \left\langle \gamma_{0,1}^2 \right\rangle_{\beta,0} + \mathcal{O}(\varepsilon).$$

We are left with the second term in (4.5). Using again Lemma 4.1 we have

$$\begin{aligned}
&\sum_y \ll j_{0,1}, (-L_0)^{-1} j_{y-1,y} (\partial_{e_y} - \partial_{e_{y-1}}) w_\lambda \gg_{\beta,\varepsilon} \\
&= \sum_{x,y} \langle j_{x-1,x}, (-L_0)^{-1} j_{y-1,y} (\partial_{e_y} - \partial_{e_{y-1}}) w_\lambda \rangle_{\beta,\varepsilon} \\
&= \sum_x \langle j_{x-1,x}, (-L_0)^{-1} j_{x-1,x} (\partial_{e_x} - \partial_{e_{x-1}}) w_\lambda \rangle_{\beta,\varepsilon} + \mathcal{O}(\varepsilon) \\
&= \langle j_{0,1}, (-L_0)^{-1} j_{0,1} (\partial_{e_1} - \partial_{e_0}) \Gamma_{w_\lambda} \rangle_{\beta,\varepsilon} + \mathcal{O}(\varepsilon) \\
&= \langle \gamma_{0,1}^2 (\partial_{e_1} - \partial_{e_0}) \Gamma_{w_\lambda} \rangle_{\beta,\varepsilon} + \mathcal{O}(\varepsilon) = \langle \gamma_{0,1} D_{0,1} \Gamma_{w_\lambda} \rangle_{\beta,\varepsilon} + \mathcal{O}(\varepsilon)
\end{aligned}$$

Even though $D_{0,1}\Gamma_{w_\lambda}$ is not a local function it is reasonable to assume that

$$\langle \gamma_{0,1} D_{0,1} \Gamma_{w_\lambda} \rangle_{\beta,\varepsilon} = \langle \gamma_{0,1} D_{0,1} \Gamma_{w_\lambda} \rangle_{\beta,0} + o_\varepsilon(1) \tag{4.9}$$

with $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$ uniformly in λ.[2] In the particular case when the local potential $W = 0$, (4.9) can actually be proven, because in this case functions g of the only energies are functions only of the modulus of the velocities, and then we have $\langle g \rangle_{\beta,\varepsilon} = \langle g \rangle_{\beta,0}$.

Hence,

$$\begin{aligned} \lim_{\lambda\to 0} \ll j_{0,1}, v_{\lambda,1} \gg_{\beta,\varepsilon} &= \left\langle \gamma_{0,1}^2 \right\rangle_{\beta,0} + \lim_{\lambda \to 0} \langle \gamma_{0,1} D_{0,1} \Gamma_{w_\lambda} \rangle_{\beta,0} + o_\varepsilon(1) \\ &= \left\langle \gamma_{0,1}^2 \right\rangle_{\beta,0} - \left\langle \eta^2 \right\rangle_{\beta,0} + o_\varepsilon(1) \\ &= \left\langle \gamma_{0,1}^2 \right\rangle_{\beta,0} + \left\langle \eta^2 \right\rangle_{\beta,0} + 2 \left\langle \gamma_{0,1}\eta \right\rangle_{\beta,0} + o_\varepsilon(1) \\ &= \left\langle (\gamma_{0,1}+\eta)^2 \right\rangle_{\beta,0} + o_\varepsilon(1). \end{aligned} \tag{4.10}$$

Hence

$$\kappa(\varepsilon,\varsigma) = \varepsilon^2 \left\langle (\gamma_{0,1}+\eta)^2 \right\rangle_{\beta,0} + o(\varepsilon^2).$$

We conclude that

$$\kappa_2(\varsigma) = \left\langle (\gamma_{0,1}+\eta)^2 \right\rangle_{\beta,0} \;\geq\; 0. \tag{4.11}$$

It follows from the above calculation that

$$\kappa_2(\varsigma) = \left\langle \gamma_{0,1}^2 \right\rangle_{\beta,0} - \sum_{x\in\mathbb{Z}} \int \left[\alpha_{0,1} \, (-\mathcal{L})^{-1} \alpha_{x,x+1} \right] \, d\rho_\beta(\mathbf{e}). \tag{4.12}$$

The right hand side of (4.12) is **exactly** the macroscopic diffusion of the energyin the autonomous stochastic dynamics describing the evolution of $\mathbf{e}$, obtained in the weak coupling limit [12, 22, 23]. Thus even if (4.12) is obtained from a formal expansion it is a mathematically well defined object and we expect it to coincide with the $\lim_{\varepsilon\to 0} \varepsilon^{-2}\kappa(\varepsilon,\varsigma)$.

4.1. **Lower bounds on $\kappa_2(\varsigma)$.** Notice that $\langle \gamma_{0,1}^{-1} \eta \rangle_\beta = 0$, so

$$1 = \langle \gamma_{0,1}^{-1}(\gamma_{0,1}+\eta) \rangle \;\leq\; \langle \gamma_{0,1}^{-2} \rangle_\beta^{1/2} \langle (\gamma_{0,1}+\eta)^2 \rangle_\beta^{1/2}$$

In particular using the last line of (4.11)

$$\left\langle \gamma_{0,1}^{-2} \right\rangle_\beta^{-1} \;\leq\; \kappa_2(\varsigma) \;\leq\; \left\langle \gamma_{0,1}^2 \right\rangle_\beta \tag{4.13}$$

In [22], a single particle Hamiltonian of the form $\mathcal{H} = p^2/2 + W(q)$ is considered in dimension $d = 2$. It is shown there under suitable assumptions on the potentials V and W that the bound $\gamma^2(e_0, e_1) \geq c_-(\varsigma) e_0 e_1$ holds, for small energies and $c_-(\varsigma) > 0$ for $\varsigma > 0$. It follows that the lower bound (4.13) is strictly positive as soon as $\varsigma > 0$ for

[2] This would follow, for example, from a uniform bound on the L^2 norm, with respect to $\langle \cdot \rangle_{\beta,\varepsilon}$, of $D_{0,1}\Gamma_{w_\lambda}$. Unfortunately, we can prove such a bound only with respect to $\langle \cdot \rangle_{\beta,0}$.

that system. We conjecture that this holds in general for $\varsigma > 0$ and we prove it for the examples of sections 6.

When the Hamiltonian part of the cell dynamics is given by a geodesic flow on a manifold of negative curvature, the lower bound in (4.13) is strictly positive even without the noise ($\varsigma = 0$), in dimension $d \geq 3$ ([12]).

5. The non-equilibrium stationary state

A different more direct way than the GK formula to study the energy flux in a macroscopic system is to consider the stationary state in a finite open system with Langevin thermostats at the boundary with temperature T and $T + \delta T$ [3]. The generator of the dynamics is then

$$L_{\varepsilon,N,\delta T} = \sum_{x=1}^{N} (A_0^x + \gamma S^x) + \varepsilon G + B_{1,T+\delta T} + B_{N,T}$$

where $B_{1,T+\delta T}, B_{N,T}$ are the generators of the corresponding Langevin dynamics at the boundaries:

$$B_{x,T} = \frac{T}{2} \partial_{p_x}^2 - p_x \partial_{p_x}$$

and

$$G = \sum_{x=2}^{N} \nabla V(q_x - q_{x-1})(\partial_{p_x} - \partial_{p_{x-1}}) \ .$$

Our goal is to compute the thermal conductivity of the stationary state, e.g. the stationary current divided by the temperature gradient $\delta T/N$:

$$\kappa_{N,T,\varepsilon} = \lim_{\delta T \to 0} \frac{N}{\delta T} \ \varepsilon \left\langle j_{0,1} \right\rangle_{N,\delta T,\varepsilon} , \tag{5.1}$$

where $< \cdot >_{N,\delta T,\varepsilon}$ is the expectation with respect to the stationary measure. To this end we are going to expand the stationary measure in ε and δT.

As a preliminary step, we use as a reference measure the inhomogeneous Gibbs distribution with linear profile of inverse temperature $\{\beta_x\}_{x=1,\dots,N}$, interpolating between the two inverse temperatures by setting $\beta_{x+1} - \beta_x \sim -\frac{\delta T}{NT^2}$. We will call $\mathbb{E}$ the expectation with respect to such a measure, that is

$$\mathbb{E}(f) = Z^{-1} \int e^{-\sum_{x=1}^{N} \beta_x e_x^\varepsilon} f(q,p) dq dp, \tag{5.2}$$

where as before $e_x^\varepsilon = \frac{1}{2}p_x^2 + \frac{1}{2}\varepsilon[V(q_x - q_{x-1}) + V(q_{x+1} - q_x)]$, for $x = 2, \dots, N-1$ and $e_1^\varepsilon = \frac{1}{2}p_1^2 + \frac{1}{2}\varepsilon V(q_2 - q_1)$, $e_N^\varepsilon = \frac{1}{2}p_N^2 + \frac{1}{2}\varepsilon V(q_N - q_{N-1})$.[3] To keep consistency with previous notations, we will use e_x to designate e_x^0, the internal energy of the isolated cell.

The corresponding adjoint operator is

$$L^*_{\varepsilon,N,\delta T} = \sum_{x=1}^{N}(-A_0^x + \gamma S^x) - \varepsilon G + \varepsilon \sum_{x=1}^{N-1}(\beta_{x+1} - \beta_x) j_{x,x+1} + B_{1,T+\delta T} + B_{N,T}.$$

We assume that there exists a unique stationary probability distribution with smooth density. The existence and uniqueness of such a probability measure still remains an open problem for most of the dynamics that appear in this work, though for some models, proofs can be found in [3] (see also [24]). For certain choice of the local dynamics L_0 and interaction V, the smoothness of the density follows by applying results of [13], [8].

Let $f_{\varepsilon,N,\delta T}$ be the density of this stationary measure with respect to this inhomogeneous Gibbs measure, i.e. the solution of

$$L^*_{\varepsilon,N,\delta T} f_{\varepsilon,N,\delta T} = 0, \quad f_{\varepsilon,N,\delta T} \geq 0.$$

If $\mathbf{e} = \{e_2, \dots, e_{N-1}\}$, then it is convenient to define the projector[4], since it integrates with respect to a gaussian the variable at the boundary.

$$\Pi f(e_2, \dots, e_{N-1}) = \mathbb{E}(f \mid \mathbf{e}) \ .$$

Also let $B = B_{1,T} + B_{N,T}$ and $J = \frac{1}{NT^2}\sum_{x=1}^{N-1} j_{x,x+1}$. We expand the stationary measure as follows

$$f_{\varepsilon,N,\delta T} = 1 + \delta T \left[w_0 + \sum_{n \geq 1} (v_n + w_n)\, \varepsilon^n \right] + \mathcal{O}((\delta T)^2) \tag{5.3}$$

where $\Pi w_n = w_n$ and $\Pi v_n = 0$. Next, it is convenient to set

$$L_B = \sum_{x=1}^{N}(A_0^x + \gamma S^x) + B = L_{0,N,0}.$$

Note that[5]

$$L_B^* = \sum_{x=1}^{N}(-A_0^x + \gamma S^x) + B = L^*_{0,N,0}.$$

[3]Since we will compute a correction of order one, the correction to the local energies does not matter.

[4]Note that this projector is different from the one used in section 3

[5]Here the adjoint is taken with respect to all the measures $\mathbb{E}(\cdot \mid \mathbf{e})$.

and that $L_B^*\Pi = \Pi L_B^* = 0$. Since $L^*_{\varepsilon,N,\delta T}\mathbf{1} = J\delta T + \mathcal{O}((\delta T)^2)$, if we compute at the first order in δT we have

$$-\varepsilon J - \varepsilon G w_0 + \sum_{n\geq 1} \varepsilon^n \left\{ L_B^* v_n - \varepsilon G v_n - \varepsilon G w_n \right\} = 0.$$

From the above it follows

$$\begin{aligned} L_B^* v_1 &= J + G w_0 \\ L_B^* v_{n+1} &= G w_n + G v_n. \end{aligned}$$

Since $\Pi G \Pi = 0$ it must be $\Pi G v_n = 0$. It is then natural to define

$$\mathcal{L}_B = \Pi G (L_B^*)^{-1} G \Pi. \tag{5.4}$$

We then obtain

$$\begin{aligned} w_0 &= \mathcal{L}_B^{-1} \Pi G (-L_B^*)^{-1} J \\ v_{n+1} &= (L_B^*)^{-1} \left[G w_n + G v_n \right] \end{aligned} \tag{5.5}$$

with $v_0 = 0$.

Next, we want to compute how $\mathcal{L}_B$ acts on the space of function $\{f \ : \ \Pi f = f\}$.

$$\begin{aligned} Gf &= \sum_{x=2}^{N} \nabla V(q_x - q_{x-1})(p_x \partial_{e_x} - p_{x-1}\partial_{e_{x-1}}) f \\ &= \sum_{x=2}^{N} j_{x,x-1}(\partial_{e_x} - \partial_{e_{x-1}}) f - \frac{1}{2}\sum_{x=2}^{N} [L_B^* V(q_x - q_{x-1})](\partial_{e_x} + \partial_{e_{x-1}}) f. \end{aligned}$$

Thus, given two function of the energies $f(e_2,\dots,e_{N-1})$ and $g(e_2,\dots,e_{N-1})$, we have[6]

$$\begin{aligned} \mathbb{E}_\beta(g\mathcal{L}_B f) &= \mathbb{E}_\beta(g \Pi G (L_B^*)^{-1} G \Pi f) \\ &= \sum_{x=2}^{N} \mathbb{E}_\beta(g G (L_B^*)^{-1} j_{x,x-1}(\partial_{e_x} - \partial_{e_{x-1}}) f), \end{aligned} \tag{5.6}$$

where we have used the antisymmetry in p of the measure. Also, taking the adjoint with respect to $\mathbb{E}_\beta$ yields

$$G^* = -G + \beta \sum_x \nabla V(q_x - q_{x-1})(p_x - p_{x-1}) = -G + \beta L_B V. \tag{5.7}$$

Inserting the above in (5.6) and using again the antisymmetry in p we have

$$-\mathbb{E}_\beta(g\mathcal{L}_B f) = \frac{1}{4}\sum_{x,y=2}^{N} \left\langle (j_{y,y-1}(\partial_{e_y} - \partial_{e_{y-1}}) g \cdot (-L_B^*)^{-1} j_{x,x-1}(\partial_{e_x} - \partial_{e_{x-1}}) f \right\rangle.$$

[6]By $\mathbb{E}_\beta$ we mean the measure (5.2) with $\delta T = 0$.

Finally, we have

$$
\begin{aligned}
\gamma(e_{x-1}, e_x)^2 \delta_{xy} &= \mathbb{E}_\beta(j_{y,y-1}(-L_B^*)^{-1} j_{x,x-1} \mid \mathbf{e}) \qquad x \neq 2, N \\
\gamma(e_1, e_2)^2 \delta_{y,2} &= \mathbb{E}_\beta(j_{y,y-1}(-L_B^*)^{-1} j_{2,1} \mid \mathbf{e}) \\
\gamma(e_{N-1}, e_N)^2 \delta_{y,N} &= \mathbb{E}_\beta(j_{y,y-1}(-L_B^*)^{-1} j_{N,N-1} \mid \mathbf{e})
\end{aligned}
$$

Thus

$$
\begin{aligned}
-\mathbb{E}_\beta(g\mathcal{L}_B f) &= \frac{1}{4}\sum_{x=2}^{N} \mathbb{E}_\beta(\gamma^2(e_{x-1}, e_x)(\partial_{e_x} - \partial_{e_{x-1}})g \cdot (\partial_{e_x} - \partial_{e_{x-1}})f) \\
&= \frac{1}{4}\sum_{x=3}^{N-1} \mathbb{E}_\beta(\gamma^2(e_{x-1}, e_x)(\partial_{e_x} - \partial_{e_{x-1}})g \cdot (\partial_{e_x} - \partial_{e_{x-1}})f) \\
&\quad + \frac{1}{4}\mathbb{E}_\beta(\gamma^2(e_2)\partial_{e_2} g \cdot \partial_{e_1,e_2} f) \\
&\quad + \frac{1}{4}\mathbb{E}_\beta(\gamma^2(e_{N-1}, e_N)\partial_{e_{N-1}} g \cdot \partial_{e_{N-1}} f)
\end{aligned}
$$

Which shows that $\mathcal{L}_B$ is the operator that one would expect in [22, 12] when adding the appropriate boundary terms.

We can, at last, compute the current:

$$
\mathbb{E}(f_{\varepsilon,N,\delta T} J) = \delta T\, \mathbb{E}(\{w_0 + \varepsilon(v_1 + w_1)\} J) + \mathcal{O}(\varepsilon^2 \delta T + (\delta T)^2).
$$

Thus, setting

$$
\mathbf{j}_{0,N} = \lim_{\delta T \to 0} \frac{1}{\delta T} \mathbb{E}(f_{\varepsilon,N,\delta T} J)
$$

we have

$$
\begin{aligned}
\mathbf{j}_{0,N} &= \varepsilon \mathbb{E}_\beta(v_1 J) + \mathcal{O}(\varepsilon^2) \\
&= \frac{\varepsilon}{N^2 T^2} \sum_x \mathbb{E}_\beta(\gamma(e_{x-1}, e_x)^2) + \varepsilon \mathbb{E}_\beta(J \cdot (L_B^*)^{-1} G w_0) + \mathcal{O}(\varepsilon^2) \\
&= \frac{\varepsilon}{N^2 T^2} \sum_x \mathbb{E}_\beta(\gamma(e_{x-1}, e_x)^2) \\
&\qquad + \frac{\varepsilon}{N} \sum_x \mathbb{E}_\beta(\alpha(e_{x-1}, e_x)[(-\mathcal{L}_B)^{-1}\alpha(e_0, e_1)]) + \mathcal{O}_N(\varepsilon^2)
\end{aligned}
$$

Formally the limit

$$
\lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{1}{\varepsilon} N \mathbf{j}_{0,N}
$$

yields the formula for κ_2 in agreement with the Green-Kubo formula expansion of Section 4.

6. Behavior of $\kappa_2(\varsigma)$ in the limit $\varsigma \to 0$ for some model systems

We now study the behavior of $\kappa_2(\varsigma)$ in the deterministic limit $\varsigma \to 0$. This limit is singular, since the operator $\mathcal{L} = \Pi G(-L_0)^{-1} G \Pi$ formally vanishes at $\varsigma = 0$ for the whole class of systems considered in this work[7]: both operators $(-L_0)^{-1}$ and G exchange symmetric and antisymmetric functions under the operation $p \to -p$, while Π annihilates antisymmetric functions. It is therefore important to analyze some particular cases in more detail. Here we consider three such examples. In all these cases, the uncoupled cells are one-dimensional and the stochasticity is the random velocity flip with rate ς^{-1}.

1. Anharmonic oscillators. It is a common belief, based on extensive numerical simulation, that transport of energy in anharmonic one-dimensional pinned chains is diffusive [21][9] (see also [17] for physical approaches passing through a kinetic limit). However, to our knowledge, there are no rigorous mathematical arguments supporting this. We show here that $\limsup_{\varsigma \to 0} \kappa_2(\varsigma) < \infty$ for one-dimensional oscillators with rather generic pinning potentials W and interaction V.

We consider the Hamiltonian (6.1) below which allows for an explicit description. The fact that as $\varsigma \to 0$, $\kappa_2(\varsigma)$ does not diverge results from averaging oscillations in the uncoupled cells, and not from decay of correlations as it would be the case for a chaotic dynamics. The control of the time integrated current-current correlations in the limit $\varsigma \to 0$ is possible if resonances between near atoms occur with small probability in the Gibbs state. This condition is violated if the pinning W is harmonic, but is otherwise typically satisfied.

2. Disordered oscillators and rotor. We next consider in more details two examples of chains of one dimensional systems that display a similar structure: the disordered harmonic chain and the rotor model. In each case, the atoms are one-dimensional systems, so that, when both noise and coupling are removed, the full dynamics becomes again integrable. Moreover, then, neighboring particles typically oscillate at different frequencies. For these two examples, we are able to give explicit formulas for the weak coupling operator $\mathcal{L}$ (see Proposition 6.3 and Proposition 6.5).

In the absence of noise ($\varsigma = 0$), the disordered chain is well known to be a perfect insulator: $\kappa = 0$ [2], while it is conjectured that the conductivity of the rotor chain is finite and positive [21], but decays

[7] When the dynamics of individual cells is chaotic, the operator $(-L_0)^{-1}$ is actually not even well defined at $\varsigma = 0$. In this Section, we will only be concerend with integrable isolated dynamics.

faster than any power law in ε as $\varepsilon \to 0$ [11]. Thus in these two cases it is expected that the conductivity of the deterministic system $\kappa(\epsilon, \varsigma)$ has no expansion in power of ϵ. What we are actually able to prove is that

$$\lim_{\varsigma \to 0} \kappa_2(\varsigma) = 0.$$

We also show in Subsection 6.3 that for the rotor chain $\limsup_{\varsigma \to 0} \kappa_2(\varsigma) < +\infty$, extending the conclusions of Proposition 6.1 to this case.

6.1. **Upper bound on the conductivity for pinned anharmonic oscillators.** Let

$$\begin{aligned} H(q,p) &= \sum_x \frac{p_x^2}{2} + W(q_x) + \varepsilon V(q_{x+1} - q_x) \\ &= \sum_x H_0(q_x, p_x) + \varepsilon V(q_{x+1} - q_x) \end{aligned} \tag{6.1}$$

with $(q_x, p_x) \in \mathbb{R}^2$. The potential W is assumed to be smooth, strictly convex, except possibly at the origin, and symmetric. The potential V is also taken smooth, symmetric, bounded below, and of polynomial growth, always satisfying the requirement that $\mu_{\beta,\varepsilon}$ is analytic in ε for small ε. To make things simple and concrete, we will actually focus on W given by

$$W(q) = \frac{|q|^r}{r}, \qquad r > 2. \tag{6.2}$$

Proposition 6.1. *Let W be given by (6.2) for some $r > 2$. Then, with the assumptions on V given after (6.1), $\limsup \kappa_2(\varsigma) < +\infty$.*

Proof. Because of its length, the proof as well as the needed introductory material are deferred to Appendix A. □

Remark. In Proposition 6.1, we have limited ourselves to a case leading to rather clean computations. A closer look at the proof in appendix A shows that our hypotheses are too restrictive: what is important is that the map $\omega(I)$, giving the frequency of oscillation as a function of the action, can be inverted. The main advantage of taking the W given by (6.2), is that this can be done explicitely.

It then arises as a natural question whether the proof could be further generalized to cases where $\omega(I)$ is invertible everywhere but on a finite or countable number of points. This would for example be the case if we consider the pinning potential $W(q) = q^2 + a\cos(q)$ for some small enough constant $a > 0$. This is unfortunatly not the case, as some logarithmic divergence in ς shows up in the limit $\varsigma \to 0$, if one just tries to mimic the proof of Proposition 6.1. Unless the system posseses some hidden symmetry, this in fact means that $\langle \gamma^2(e_0, e_1)\rangle_{\beta,0}$ diverges logarithmically in the deterministic limit. This however does not necessarily imply that κ_2 itself will diverge in this limit, as the term $\langle \eta^2 \rangle_{\beta,0}$ in (4.11) can compensate this divergence. This is in fact what is expected to happen.

6.2. **The disordered harmonic chain.** The hamiltonian part of the generator is now given by

$$A_0 \;=\; \sum_x p_x \partial_{q_x} - \omega_x^2 q_x \partial_{p_x}, \qquad G \;=\; \sum_x (q_{x-1} - 2q_x + q_{x+1})\partial_{p_x}, \tag{6.3}$$

where ω_x^2 are random, independent and identically distributed squared frequencies, that satisfy the bound $c^{-1} \leq \omega_x^2 \leq c$, for some constant $c > 0$. The internal energy is given by $e_x = p_x^2/2 + \omega_x^2 q_x^2/2$, while for $\varepsilon \geq 0$ the energy flux $\varepsilon j_{x,x+1}$ between two adjacent oscillators is given by

$$\varepsilon\, j_{x,x+1} \;=\; -\varepsilon\, \frac{p_x + p_{x+1}}{2}(q_{x+1} - q_x).$$

Lemma 6.2. *Let $x, y \in \mathbb{Z}$. A solution $\psi_{x,y}$ to the equation*

$$-L_0 \psi_{x,y} \;=\; q_x p_y$$

is given by

$$\psi_{x,y} = \frac{4\varsigma\left(\omega_x^2 q_x p_y - \omega_y^2 q_y p_x\right) + (\omega_x^2 - \omega_y^2)p_x p_y + \left((\omega_x^2 - \omega_y^2 - 8\varsigma^2)\omega_y^2\right) q_x q_y}{\Delta(x,y)}$$

with

$$\Delta_{x,y} \;=\; 8\varsigma^2(\omega_x^2 + \omega_y^2) + (\omega_x^2 - \omega_y^2)^2.$$

Proof. This follows by a direct computation. □

This lemma allows us to give an explicit form of the operator $\mathcal{L} = \Pi G (L_0)^{-1} G \Pi$. We know that $\mathcal{L}$ is the generator of a Ginzburg-Landau dynamics.

Proposition 6.3. *Let $\mathcal{L} = \Pi G (L_0)^{-1} G \Pi$. Then*

$$\rho_\beta(\mathbf{e}) \;=\; \prod_x \left(\beta \mathrm{e}^{-\beta e_x}\right), \tag{6.4}$$

$$\gamma^2(e_x, e_{x+1}) \;=\; \frac{4\varsigma}{\Delta_{x,x+1}} e_x e_{x+1}, \tag{6.5}$$

$$\alpha(e_x, e_{x+1}) \;=\; \frac{8\varsigma}{\Delta_{x,x+1}}(e_x - e_{x+1}) \tag{6.6}$$

Proof. To obtain the expression for the invariant measure, let us take an f that depends only on $e_x = (p_x^2 + \omega_x^2 q_x^2)/2$, and let us compute

$$\begin{aligned} \langle f \rangle_{\beta,0} &= Z_x(\beta)^{-1} \int_{\mathbb{R}^2} f\Big(\frac{p_x^2 + \omega_x^2 q_x^2}{2}\Big)\, \mathrm{e}^{-\beta(p_x^2+\omega_x^2 q_x^2)/2}\, \mathrm{d}q_x \mathrm{d}p_x \\ &\sim \int_0^\infty f(e)\, \mathrm{e}^{-\beta e} \mathrm{d}e \end{aligned}$$

from which the expression for ρ_β follows.

Next we have that

$$\begin{aligned} \gamma^2(e_x, e_{x+1}) &= \Pi\big(j_{x,+1}(-L_0)^{-1} j_{x,x+1}\big) \\ &= \frac{1}{4}\Pi\Big(q_{x+1}p_x - q_x p_{x+1} + q_{x+1}p_{x+1} - q_x p_x\Big) \\ &\qquad \Big(\psi_{x+1,x} - \psi_{x,x+1} + \psi_{x+1,x+1} - \psi_{x,x}\Big) \\ &= \frac{2\sigma}{\Delta_{x,x+1}}\Pi\Big(\omega_{x+1}^2 q_{x+1}^2 p_x^2 + \omega_x^2 q_x^2 p_{x+1}^2\Big) \end{aligned}$$

where we have used the fact that odd powers of $q_x, p_x, q_{x+1}, p_{x+1}$ are annihilated by the projection Π. Using then polar coordinates

$$\frac{\omega_x q_x}{\sqrt{2}} = \sqrt{e_x}\cos\theta_x, \qquad \frac{p_x}{\sqrt{2}} = \sqrt{e_x}\sin\theta_x,$$

it is computed that both

$$\Pi(p_x^2) = \Pi(\omega_x^2 q_x^2) = \frac{1}{2\pi}\int_0^{2\pi} 2e\sin^2\theta_x\, \mathrm{d}\theta_x = e_x.$$

This yields the announced expression for $\gamma^2(e_x, e_{x+1})$.

The current $\alpha(e_x, e_{x+1})$ follows using (3.10). □

Corollary 6.4. *For $\varsigma > 0$, we have that a.s. in ω*

$$\kappa_2(\varsigma) = \frac{8\varsigma}{\langle \Delta_{0,1}(\varsigma)\rangle_*} > 0$$

where $\langle\cdot\rangle_*$ represents the average with respect to the realizations of the disorder. In particular, a.s. in ω,

$$\lim_{\varsigma\to 0}\kappa_2(\varsigma) = 0.$$

Proof. The proof is given in Appendix B. □

6.3. **The rotor chain.** The Hamiltonian part of the dynamics is given by

$$A_0 \;=\; \sum_x p_x \partial_{q_x}, \qquad G \;=\; \sum_x \left[\sin(q_{x-1}-q_x) - \sin(q_x - q_{x+1})\right] \partial_{p_x}, \tag{6.7}$$

with $q_x \in \mathbb{R}/2\pi\mathbb{Z}$. The individual energy for the uncoupled dynamics ($\varepsilon = 0$) is $e_x = p_x^2/2$. If $\varepsilon > 0$, there is a flux of energy which is given by $\varepsilon j_{x,x+1}$ where

$$j_{x,x+1} = -\frac{1}{2}(p_x + p_{x+1}) \sin(q_{x+1} - q_x)$$

Proposition 6.5. *For this system*

$$\rho(\mathbf{e}) \;=\; \prod_x \left(\mathrm{e}^{-(U(e_x)+\beta e_x)} \sqrt{\beta/\pi} \right) \quad \textit{with} \quad U(e_x) \;=\; \frac{1}{2}\log e_x, \tag{6.8}$$

$$\gamma^2(e_x, e_{x+1}) \;=\; \frac{2\varsigma\, e_x e_{x+1}}{\Delta(e_x, e_{x+1})}, \tag{6.9}$$

$$\alpha(e_x, e_{x+1}) \;=\; \frac{\varsigma(e_x - e_{x+1})}{\Delta^2(e_x, e_{x+1})} \big(\Delta(e_x, e_{x+1}) + 8 e_x e_{x+1}\big) \tag{6.10}$$

with

$$\Delta(e_x, e_{x+1}) \;=\; 4\varsigma^2(e_x + e_{x+1}) + (e_{x+1} - e_x)^2.$$

Proof. The proof is given in Appendix C. □

It is seen from the above expressions that as noted earlier the generator $\mathcal{L}$ formally vanishes as $\varsigma \to 0$. However, for ς small but positive, the coefficient $\gamma^2(e_x, e_{x+1})$ can become of order $1/\varsigma$ in case a resonance occurs, such that $|e_{x+1} - e_x| \leq \varsigma$. We have unfortunately not been able to decide whether, despite of this phenomenon, the value of $\kappa_2(\varsigma)$ still vanishes as $\varsigma \to 0$, as suggested by the results in [11].

We have however a result analogous to that of Proposition 6.1:

Proposition 6.6. *For any $\varsigma > 0$, $\kappa_2(\beta, \varsigma)$ is strictly positive and*

$$\limsup_{\varsigma \to 0} \kappa_2(\varsigma) < +\infty.$$

Proof. By (4.13) and the explicit form of γ we have that

$$\kappa_2(\varsigma, \beta) \geq \langle \gamma^{-2}(e_0, e_1) \rangle_{\beta,0}^{-1} \geq c\varsigma$$

for a positive constant c independent of ς. By (4.13) it holds also that

$$\kappa_2 \;\leq\; \langle \gamma^2(e_0, e_1) \rangle_{\beta,0}.$$

The function $\langle \gamma^2(e_0,e_1)\rangle_{\beta,0}$ has the behavior

$$\begin{aligned}
\langle \gamma^2(e_0,e_1)\rangle_{\beta,0} &\sim \int_{\mathbb{R}^2_+} \frac{\varsigma\, e_0 e_1}{4\varsigma^2(e_0+e_1)+(e_1-e_0)^2} \mathrm{e}^{-\beta(e_0+e_1)} \frac{\mathrm{d}e_0 \mathrm{d}e_1}{\sqrt{e_0 e_1}} \\
&\sim \int_{\mathbb{R}^2} \frac{\varsigma\, x^2 y^2}{8\varsigma^2(x^2+y^2)+(y^2-x^2)^2} \mathrm{e}^{-\beta(x^2+y^2)/2}\, \mathrm{d}x\mathrm{d}y \\
&\sim \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, \frac{\varsigma\, r^3 \cos^2\theta \sin^2\theta}{8\varsigma^2 + r^2(\cos^2\theta - \sin^2\theta)^2}\, \mathrm{e}^{-\beta r^2/2}.
\end{aligned}$$

In the limit $\varsigma \to 0$, only the values of θ such that $\cos^2\theta - \sin^2\theta \sim 0$ contribute ($\theta \sim \pm\pi/4$ and $\theta \sim \pm 3\pi/4$), so that, by a Taylor expansion,

$$\begin{aligned}
\langle \gamma^2(e_0,e_1)\rangle_{\beta,0} &\sim \int_0^\infty \mathrm{d}r \int_0^1 du\, \frac{\varsigma\, r^3}{8\varsigma^2 + r^2 u^2}\, \mathrm{e}^{-\beta r^2/2} \\
&\sim \int_0^\infty r^2 \mathrm{e}^{-\beta r^2/2} \left(\int_0^1 \frac{\varsigma/r}{8(\varsigma/r)^2+u^2}\, \mathrm{d}u \right) \mathrm{d}r \\
&\sim 1 \quad \text{as} \quad \varsigma \to 0.
\end{aligned}$$

This proves the claim. □

Appendix A. Proof of Proposition 6.1

To study the system at hand, it is convenient to pass to action-angle variables. Let $\mathrm{I} : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$\mathrm{I}(E) \;=\; \frac{1}{2\pi} \int_{A(E)} \mathrm{d}q\mathrm{d}p \quad \text{with} \quad A(E) \;=\; \{(q,p) \in \mathbb{R}^2 : H_0(q,p) \le E\}$$

Our assumptions on W ensure that $\mathrm{I}'(E) = \mathrm{d}\mathrm{I}/\mathrm{d}E\,(E) > 0$ for any $E > 0$. Given $E \ge 0$, we also set

$$q^*(E) \;=\; \max\{q \in \mathbb{R} : H_0(q,p) = E \text{ for some } p \in \mathbb{R}\}.$$

Then we define the action-angle variables by

$$\begin{aligned}
I_x &= I(q_x,p_x) \;=\; \mathrm{I}(H_0(q_x,p_x)), \\
\theta_x &= \theta(q_x,p_x) \;=\; \frac{-\mathrm{sgn}(p_x)}{\mathrm{I}'(H_0(q_x,p_x))} \int_{q_x}^{q^*(H_0(q_x,p_x))} \frac{\mathrm{d}q'}{\sqrt{2(H_0(q_x,p_x)-W(q'))}}.
\end{aligned}$$

It is checked that $(I_x,\theta_x) \in \mathbb{R}_+ \times \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The potential W is such that this change of variable is invertible, except at origin. We denote by Q and P the inverse maps:

$$q_x = Q(I_x,\theta_x), \qquad p_x = P(I_x,\theta_x).$$

The change of variables $(q_x,p_x) \leftrightarrow (I_x,\theta_x)$ is known to be a canonical change of variables.

Let $\mathrm{H}_0 : \mathbb{R}_+ \to \mathbb{R}_+$ be the inverse function of I: $\mathrm{H}_0 \circ \mathrm{I}(E) = E$ for any $E \in \mathbb{R}_+$. In the action-angle variables, the Hamiltonian (6.1) reads

$$H(I,\theta) \;=\; \sum_x \mathrm{H}_0(I_x) + \epsilon V\big(Q(I_{x+1},\theta_{x+1}) - Q(I_x,\theta_x)\big).$$

Defining

$$\omega(I_x) \;=\; \mathrm{H}_0'(I_x) \;=\; \mathrm{d}\mathrm{H}_0/\mathrm{d}I_x,$$

Hamilton equations read

$$\begin{aligned}\dot I_x \;&=\; -\epsilon\frac{\partial}{\partial\theta_x}V\big(Q(I_{x+1},\theta_{x+1}) - Q(I_x,\theta_x)\big),\\ \dot\theta_x \;&=\; \omega_x \;+\; \epsilon\frac{\partial}{\partial I_x}V\big(Q(I_{x+1},\theta_{x+1}) - Q(I_x,\theta_x)\big).\end{aligned}$$

The current, given by (2.3), has the form

$$j_{x,x+1} \;=\; -\frac{1}{2}\big(P(I_x,\theta_x) + P(I_{x+1},\theta_{x+1})\big)V'\big(Q(I_{x+1},\theta_{x+1}) - Q(I_x,\theta_x)\big) \tag{A.1}$$

with $V'(x) = \mathrm{d}V/\mathrm{d}x$.

Since we are in dimension $d = 1$, the noise written in the action-angle coordinates is given by

$$Sf(I,\theta) \;=\; \sum_x \big(f(I,\theta^x) - f(I,\theta)\big), \tag{A.2}$$

with θ^x is obtained from θ by changing θ_x to $-\theta_x$ ($-\theta_x$ is the inverse of θ_x for the the addition on $\mathbb{T}$). The symmetry of the potential W implies

$$P(I_x,-\theta_x) \;=\; -P(I_x,\theta_x) \qquad \text{and} \qquad Q(I_x,-\theta_x) \;=\; Q(I_x,\theta_x).$$

This implies that the noise S, as defined by (A.2), preserves the total energy, and that the relation

$$Sj_{x,x+1} = -4j_{x,x+1}$$

holds.

A.1. **The special case W given by (6.2).** Let us now assume that $W(q) = |q|^r/r$, i.e.

$$H_0(q,p) \;=\; \frac{p^2}{2} + \frac{|q|^r}{r}, \qquad r > 2. \tag{A.3}$$

The following scaling relation is checked thanks to Hamilton's equations: if $(q(t),p(t))_{t\ge 0}$ is a solution to the equations of motion, then so is $(q_\alpha(t),p_\alpha(t))_{t\ge 0}$ with

$$q_\alpha(t) \;=\; \alpha^{2/(r-2)}q(\alpha t), \qquad p_\alpha(t) \;=\; \alpha^{r/(r-2)}p(\alpha t) \qquad \text{for} \qquad \alpha > 0.$$

This first allows to deduce that

$$\mathrm{H}_0(I) \;=\; \mathrm{H}_0(1)\cdot I^{2r/(r+2)} \qquad \text{and} \qquad \omega(I) \;=\; \omega(1)\cdot I^{(r-2)/(r+2)}. \tag{A.4}$$

Next, writing

$$Q(I,\theta) \;=\; \sum_{k\in\mathbb{Z}} \hat{Q}(I,k)\mathrm{e}^{\mathrm{i}k\theta}, \qquad P(I,\theta) \;=\; \sum_{k\in\mathbb{Z}} \hat{P}(I,k)\mathrm{e}^{\mathrm{i}k\theta}, \tag{A.5}$$

we obtain

$$Q(I,\theta) \;=\; I^{2/(r+2)}\sum_{k\in\mathbb{Z}} \hat{Q}(1,k)\mathrm{e}^{\mathrm{i}k\theta}, \qquad P(I,\theta) \;=\; I^{r/(r+2)}\sum_{k\in\mathbb{Z}} \hat{P}(1,k)\mathrm{e}^{\mathrm{i}k\theta}. \tag{A.6}$$

It follows from the theory of ordinary differential equations that $Q(1,\theta)$ and $P(1,\theta)$ are smooth, so that the Fourier coefficients $\hat{Q}(1,k)$, $\hat{P}(1,k)$, with $k\in\mathbb{Z}$, have good decay property as $|k|\to\infty$.

A.2. **Poisson equation for the uncoupled dynamics.** In this subsection, we consider functions on $\mathbb{R}_+^2\times\mathbb{T}^2$, that depend on two actions (I_0,I_1) and two angles (θ_0,θ_1). The actions play the role of a parameter, and, for clarity, will be dropped from several notations. A function $f\in\mathcal{C}^\infty(\mathbb{T}^2)$ is expanded in Fourier series as

$$f(I_0,I_1,\theta_0,\theta_1) \;=\; \sum_{(k_0,k_1)\in\mathbb{Z}} \hat{f}(I_0,I_1,k_0,k_1)\mathrm{e}^{\mathrm{i}(k_0\theta_0+k_1\theta_1)}$$

with

$$\hat{f}(I_0,I_1,k_0,k_1) \;=\; \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} f(I_0,I_1,\theta_0,\theta_1)\mathrm{e}^{-\mathrm{i}(k_0\theta_0+k_1\theta_1)}.$$

It is seen that the current satisfies $\hat{j}_{0,1}(I_0,I_1,0,0)=0$ for all $(I_0,I_1)\in\mathbb{R}_+^2$. We introduce the notations

$$\begin{aligned}\alpha(k_0,k_1) \;&=\; \mathrm{i}\big(k_0\omega(I_0)+k_1\omega(I_1)\big)-2\varsigma\\ D(k_0,k_1) \;&=\; \alpha(k_0,k_1)\alpha(-k_0,-k_1)-\frac{16\varsigma^4}{\alpha(-k_0,k_1)\alpha(k_0,-k_1)}.\end{aligned}$$

Lemma A.1. *Let f be a function on $\mathbb{R}_+^2\times\mathbb{T}^2$ such that $f(I_0,I_1,\cdot,\cdot)$ is smooth and satisfies $\hat{f}(I_0,I_1,0,0)=0$, for any $(I_0,I_1)\in\mathbb{R}_+^2$. Writing $\hat{f}(k_0,k_1)$ for $\hat{f}(I_0,I_1,k_0,k_1)$, we define*

$$\begin{aligned}g(I_0,I_1,k_0,k_1) \;&=\; \hat{f}(k_0,k_1)\;-\;\varsigma\Big(\frac{\hat{f}(-k_0,k_1)}{\alpha(-k_0,k_1)}+\frac{\hat{f}(k_0,-k_1)}{\alpha(k_0,-k_1)}\Big)\\ &+\;\frac{\varsigma^2}{\alpha(-k_0,-k_1)}\Big(\frac{1}{\alpha(-k_0,k_1)}+\frac{1}{\alpha(k_0,-k_1)}\Big)\big(\hat{f}(-k_0,-k_1)-\hat{f}(k_0,k_1)\big).\end{aligned} \tag{A.7}$$

A solution u to the equation $-L_0 u = f$ is given, in the Fourier variables, by

$$\begin{aligned}\hat{u}(I_0, I_1, 0, 0) &= 0,\\ \hat{u}(I_0, I_1, k_0, k_1) &= -\frac{\alpha(-k_0, -k_1)}{D(k_0, k_1)}\, g(I_0, I_1, k_0, k_1) \quad \textit{for} \quad (k_x, k_y) \neq (0, 0).\end{aligned}$$

Proof. In the Fourier variables, the equation $-L_0 u = f$ reads

$$\alpha(k_0, k_1)\hat{u}(k_0, k_1) + \varsigma\hat{u}(-k_0, k_1) + \varsigma\hat{u}(k_0, -k_1) \;=\; -\hat{f}(k_0, k_1)$$

where we have written $\hat{u}(k_0, k_1)$ for $\hat{u}(I_0, I_1, k_0, k_1)$. The result is then checked by means of a direct computation. □

Remarks. 1. All other solutions are obtained by taking for $\hat{u}(I_0, I_1, 0, 0)$ an arbitrary function of the actions I_0, I_1. This choice is irrelevant for the sequel.
2. Since $|\varsigma/\alpha(k_0, k_1)| \leq 1$ for all $(k_0, k_1) \in \mathbb{Z}^2$, we have the bound

$$|g(I_0, I_1, k_0, k_1)| \;\leq\; 5 \max\{|\hat{f}(k_0, \pm k_1)|, |\hat{f}(-k_0, \pm k_1)|\}.$$

3. For $\varsigma = 0$, the solution simply becomes

$$\hat{u}(I_0, I_1, k_0, k_1) \;=\; \mathrm{i}\frac{\hat{f}(k_0, k_1)}{k_0\omega(I_0) + k_1\omega(I_1)} \qquad \text{for} \qquad (k_x, k_y) \neq (0, 0).$$

A.3. **Proof of Proposition 6.1.** By (4.13) we have

$$\kappa_2(\varsigma, \beta) \;\leq\; \langle \gamma^2(e_0, e_1)\rangle_{\beta,0} \;=\; \langle j_{0,1}(-L_0)^{-1} j_{0,1}\rangle_{\beta,0},$$

with $\langle\cdot\rangle_{\beta,0}$ the uncoupled Gibbs state. Writing $u = (-L_0)^{-1} j_{0,1}$ we have thus

$$\begin{aligned}\langle j_{0,1}(-L_0)^{-1} j_{0,1}\rangle_{\beta,0} \;\sim\; & \int_{\mathbb{R}^2_+} \mathrm{e}^{-\beta\left(\mathrm{H}_0(I_0)+\mathrm{H}_0(I_1)\right)}\, \mathrm{d}I_0\mathrm{d}I_1\\ & \times \int_{\mathbb{T}^2} u(I_0, I_1, \theta_0, \theta_1) j_{0,1}(I_0, I_1, \theta_0, \theta_1)\, \mathrm{d}\theta_0\mathrm{d}\theta_1\\ \sim\; & \int_{\mathbb{R}^2_+} \mathrm{e}^{-\beta\left(\mathrm{H}_0(I_0)+\mathrm{H}_0(I_1)\right)}\, \mathrm{d}I_0\mathrm{d}I_1\\ & \times \sum_{k_0,k_1\in\mathbb{Z}^2} \hat{u}(I_0, I_1, k_0, k_1)\hat{j}_{0,1}(I_0, I_1, k_0, k_1).\end{aligned}$$

Writing

$$h(I_0, I_1, k_0, k_1) \;=\; -g(I_0, I_1, k_0, k_1)\hat{j}(I_0, I_1, k_0, k_1), \tag{A.8}$$

with g as defined in (A.7), we obtain by Lemma A.1,

$$\langle j_{0,1}(-L_0)^{-1}j_{0,1}\rangle_{\beta,0} \;\sim\; \sum_{(k_0,k_1)\neq(0,0)} \int_{\mathbb{R}_+^2} \frac{\alpha(-k_0,-k_1)}{D(k_0,k_1)}\, h(I_0,I_1,k_0,k_1)\, \mathrm{e}^{-\beta\left(\mathrm{H}_0(I_0)+\mathrm{H}_0(I_1)\right)}\, \mathrm{d}I_0\mathrm{d}I_1. \tag{A.9}$$

In this expression,

$$\frac{\alpha(-k_0,-k_1)}{D(k_0,k_1)} \;=\; -\frac{2\varsigma \,+\, \mathrm{i}\left(k_0\omega(I_0)+k_1\omega(I_1)\right)}{\left(k_0\omega(I_0)+k_1\omega(I_1)\right)^2 \,+\, 4\varsigma^2\frac{(k_0\omega(I_0)-k_1\omega(I_1))^2}{(k_0\omega(I_0)-k_1\omega(I_1))^2+4\varsigma^2}}. \tag{A.10}$$

We now come to the crux of the argument, and start using the specific form of H_0. In view of (A.10), it looks desirable to change integration variables in (A.9) from (I_0, I_1) to $(\omega_0,\omega_1) = (\omega(I_0),\omega(I_1))$. The anharmonicity of W, specifically expressed in this case by relation (A.4), makes this possible, giving

$$\langle j_{0,1}(-L_0)^{-1}j_{0,1}\rangle_{\beta,0} \;\sim\; \sum_{(k_0,k_1)\neq(0,0)} \int_{\mathbb{R}_+^2} \frac{\alpha(-k_0,-k_1)}{D(k_0,k_1)}\, \tilde h(\omega_0,\omega_1,k_0,k_1)\, (\omega_0\omega_1)^{\frac{4}{r-2}}\, \rho_\beta(\omega_0,\omega_1)\, \mathrm{d}\omega_0\mathrm{d}\omega_1 \tag{A.11}$$

with

$$\begin{aligned}\tilde h\big(\omega_0,\omega_1,k_0,k_1\big) \;&=\; h\big(c(r)\omega_0^{\frac{r+2}{r-2}}, c(r)\omega_1^{\frac{r+2}{r-2}}, k_0, k_1\big), \qquad c(r)>0,\\ \rho_\beta(\omega_0,\omega_1) \;&=\; \mathrm{e}^{-c'(r)\beta(\omega_0^{\frac{2r}{r-2}}+\omega_1^{\frac{2r}{r-2}})}, \qquad c'(r)>0.\end{aligned}$$

To proceed, we need some more technical informations on the function $\tilde h(\omega_0,\omega_1,k_0,k_1)$. The potential W is not strictly convex at the origin, implying that $\omega(I)$ vanishes as $I\to 0$. For this reason, we need a relatively detailed knowledge on $\tilde h(\omega_0,\omega_1,k_0,k_1)$ for (ω_0,ω_1) near the origin, in a order to exclude any divergence at small frequencies.

Using the general expression (A.1) for the current $j_{0,1}$, the specific expression (A.6) for $Q(I,\theta)$ and $P(I,\theta)$, the definition (A.7) of g, and the definition (A.8) of h, we conclude that h is of the form

$$\begin{aligned}h(I_0,I_1,k_0,k_1) \;=\; I_0^{\frac{2r}{r+2}}h_{0,0}(I_0,I_1,k_0,k_1) + I_0^{\frac{r}{r+2}}I_1^{\frac{r}{r+2}}h_{0,1}(I_0,I_1,k_0,k_1)\\ +\; I_1^{\frac{2r}{r+2}}h_{1,1}(I_0,I_1,k_0,k_1),\end{aligned}$$

so that in turn $\tilde{h}$ takes the form

$$\tilde{h}(\omega_0, \omega_1, k_0, k_1) = \omega_0^{2r/(r-2)} \tilde{h}_{0,0}(\omega_0, \omega_1, k_0, k_1) + \omega_0^{r/(r-2)} \omega_1^{r/(r-2)} \tilde{h}_{0,1}(\omega_0, \omega_1, k_0, k_1) + \omega_1^{2r/(r-2)} \tilde{h}_{1,1}(\omega_0, \omega_1, k_0, k_1), \tag{A.12}$$

where $\tilde{h}_{i,j}$ satisfies the following bounds: there exists $a < +\infty$ and, for any $b > 0$, there exists a constant $\mathrm{C}_b < +\infty$, such that

$$\tilde{h}_{i,j}(\omega_0, \omega_1, k_0, k_1) \le \mathrm{C}_b \frac{(|\omega_0| + |\omega_1| + 1)^a}{(|k_0| + |k_1| + 1)^b}, \quad (i,j) = (0,0), (0,1), (1,1). \tag{A.13}$$

Moreover, by symmetry, we have $\hat{P}(I, 0) = 0$ for all $I > 0$, with $\hat{P}(I, 0)$ defined by (A.5). It follows that

$$\tilde{h}_{0,0}(\omega_0, \omega_1, 0, k_1) = \tilde{h}_{0,1}(\omega_0, \omega_1, 0, k_1) = \tilde{h}_{0,1}(\omega_0, \omega_1, k_0, 0) = \tilde{h}_{1,1}(\omega_0, \omega_1, k_0, 0) = 0. \tag{A.14}$$

We now move back to the evalution of (A.11). We distinguish three cases, according to the values of k_0 and k_1; resonances appear in case 3. The sum over $(k_0, k_1) \in \mathbb{Z}^2/\{0, 0\}$ can then be controlled thanks to the decay in (A.13) with b large enough.

Case 1: $k_0 k_1 = 0$. Let us, as an example, consider the case $k_0 = 0, k_1 \neq 0$. The integral (A.11) has a possible divergence only for $k_1 \to 0$. We have

$$\left| \frac{\alpha(-k_0, -k_1)}{D(k_0, k_1)} \right| \le \mathrm{C}\omega_1^{-2} \qquad \text{for} \qquad \omega_0, \omega_1 \le 1$$

Thanks to (A.14), only the term in $\tilde{h}_{1,1}$ survives in (A.12), and we conclude that the integrand behaves as

$$\omega_1^{4/(r-2)} \omega_1^{2r/(r-2)} \omega_1^{-2} = \omega_1^{8/(r-2)} \qquad \text{as} \qquad \omega_1 \to 0,$$

so that there is in fact no singularity.

Case 2: $k_0 k_1 > 0$. The only possible divergence of the integral (A.11) is at the origin. We have the bounds

$$\left| \frac{\alpha(-k_0, -k_1)}{D(k_0, k_1)} \right| \le \mathrm{C}\omega_0^{-2}, \mathrm{C}\omega_0^{-1}\omega_1^{-1}, \mathrm{C}\omega_1^{-2} \qquad \text{for} \qquad \omega_0, \omega_1 \le 1,$$

allowing to check, as in the previous case, that there is no singularity.

Case 3: $k_0 k_1 < 0$. The integrand now becomes truly singular (resonances). Let us assume, for example, that $k_0 > 0$ and $k_1 < 0$. We split the integral (A.11) as

$$\int_{\mathbb{R}^2_+} (\dots) \;=\; \int_{k_0\omega_0+|k_1|\omega_1<\varsigma} (\dots) \;+\; \int_{k_0\omega_0+|k_1|\omega_1\geq\varsigma} (\dots). \tag{A.15}$$

For the first integral, we are satisfied by the rough bound

$$\Big|\frac{\alpha(-k_0,-k_1)}{D(k_0,k_1)}\Big| \;\leq\; \frac{\mathrm{C}}{\varsigma^2(k_0\omega_0+|k_1|\omega_1)^2}.$$

As in the cases treated previousely, it is seen that there is no singularity. Moreover, the intgration domain is of size ς^2, so that the intgral is of order 1 at most.

We move to the second integral. We find it convenient to change again variables. With

$$x \;=\; k_0\omega_0 + |k_1|\omega_1, \qquad y \;=\; k_0\omega_0 - |k_1|\omega_1,$$

the second integral in the right hand side of (A.15) becomes

$$\begin{aligned}&\int_{k_0\omega_0+|k_1|\omega_1\geq\varsigma} (\dots)\\ &\quad\sim \int_{\varsigma}^{\infty} \mathrm{d}x \int_{-x}^{x} \mathrm{d}y \frac{2\varsigma+\mathrm{i}y}{y^2+4\varsigma^2\frac{x^2}{x^2+4\varsigma^2}}\, \phi(x,y,k_0,k_1)\, \tilde{\rho}_\beta(x,y,k_0,k_1)\end{aligned}$$

with

$$\begin{aligned}\phi(x,y,k_0,k_1) \;&=\; h\left(\frac{x+y}{2k_0},\frac{x-y}{2|k_1|},k_0,k_1\right)\left(\frac{(x+y)(x-y)}{4k_0|k_1|}\right)^{4/(r-2)}\\ \tilde{\rho}_\beta(x,y,k_0,k_1) \;&=\; \rho_\beta\left(\frac{x+y}{2k_0},\frac{x-y}{2|k_1|}\right).\end{aligned}$$

We observe that, in the domain of integration $x \geq \varsigma$:

$$4\varsigma^2\frac{x^2}{x^2+4\varsigma^2} \;\geq\; \frac{4}{5}\varsigma^2.$$

Therefore, the integral converges to a finite value as $\varsigma \to 0$. □

Appendix B. Proof of Corollary 6.4

Consider the quenched space-time correlations of the energy:

$$S(x,t,\omega) \;= \langle e_x(t)e_0(0)\rangle_{\rho_\beta} - \beta^{-2}$$

where $\{e_x(t)\}$ is the time evolved energy generated by the Ginzburg-Landau dynamics $\mathcal{L}$ with the coefficients γ^2 and α computed above, starting with the equilibrium distribution at temperature β^{-1}. Then computing the time derivative we have

$$\begin{aligned}\partial_t S(x,t,\omega) &= 8\varsigma\Delta^{-1}_{x+1,x,\omega}\left[S(x+1,t,\omega)-S(x,t,\omega)\right]\\ &\quad -8\varsigma\Delta^{-1}_{x,x-1,\omega}\left[S(x,t,\omega)-S(x-1,t,\omega)\right]\end{aligned}$$

i.e. $S(x,t,\omega) = \mathbb{E}_{0,\omega}(\delta_x(X(t)))$, the transition probability of a 1-dimensional random walk on random bonds $X(t)$ (so called bond diffusion). It is well known and easy to compute the asymptotic variance of this bond diffusion, it is given by the harmonic average of the bonds variables ([20]):

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\sum_x x^2 S(x,t,\omega) = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{0,\omega}(X(t)^2) &= \left\langle\left(\frac{8\varsigma}{\Delta_{0,1}(\varsigma)}\right)^{-1}\right\rangle_*^{-1}\\ &= \frac{8\varsigma}{\langle\Delta_{0,1}(\varsigma)\rangle_*}\end{aligned}\tag{B.1}$$

almost surely in ω.

By the Green-Kubo formula for the diffusivity for $\mathcal{L}$, this is equal to $\kappa_2(\varsigma,\beta)$ and

$$\kappa_2(\varsigma) = \frac{8\varsigma}{\langle\Delta_{0,1}(\varsigma)\rangle_*} \to 0 \qquad \text{as} \qquad \varsigma\to 0, \tag{B.2}$$

which gives the claims. □

Appendix C. Proof of Proposition 6.5

We start by the following lemma.

Lemma C.1. *Let $x, y \in \mathbb{Z}$. A solution $\psi_{x,y}$ to the equation*

$$-L_0\psi_{x,y} = \sin(q_x - q_y)p_x \tag{C.1}$$

is given by

$$\begin{aligned}\psi_{x,y} &= \Delta^{-1}_{x,y}\left\{[4\varsigma^2 + (e_x - e_y)]e_x + \frac{1}{2}(e_x-e_y)p_xp_y\right\}\cos(q_x-q_y)\\ &\quad + \Delta^{-1}_{x,y}\left\{2\varsigma(e_yp_x + e_xp_y)\right\}\sin(q_x-q_y)\end{aligned}\tag{C.2}$$

with

$$\Delta_{x,y} := \Delta(e_x,e_y) = 4\varsigma^2(e_x+e_y) + (e_y-e_x)^2. \tag{C.3}$$

Proof. We compute

$$\begin{aligned} A_0\psi_{x,y} &= 2\varsigma\Delta_{x,y}^{-1}(e_y p_x + e_x p_y)(p_x - p_y)\cos(q_x - q_y) \\ &- \Delta_{x,y}^{-1}\Big\{\big(4\varsigma^2 + (e_x - e_y)\big)e_x + \frac{1}{2}(e_x - e_y)p_x p_y\Big\}(p_x - p_y)\sin(q_x - q_y) \end{aligned}$$

and

$$S\psi_{x,y} = -4\varsigma\Delta_{x,y}^{-1}(e_y p_x + e_x p_y)\sin(q_x - q_y) - 2\Delta_{x,y}^{-1}(e_x - e_y)p_x p_y\cos(q_x - q_y).$$

Remembering that $p_x^2 = 2e_x$ and $p_y^2 = 2e_y$, the terms in $\cos(q_x - q_y)$ cancel in $(A_0 + \varsigma S)\psi_{x,y}$, so that

$$[A_0 + \varsigma S]\psi_{x,y} = \Delta_{x,y}^{-1}\theta_{x,y}\sin(q_x - q_y)$$

with

$$\begin{aligned} \theta_{x,y} &= \Big\{\big(4\varsigma^2 + (e_x - e_y)\big)e_x + \frac{1}{2}(e_x - e_y)p_x p_y\Big\}(p_x - p_y) \\ &\quad - 4\varsigma^2(e_y p_x + e_x p_y) \\ &= -p_x\Delta_{x,y}. \end{aligned}$$

This proves the claim. □

We now move to the proof of Proposition 6.5. The Gibbs measure at inverse temperature β is readily computed. For a function f depending only on the uncoupled energy $e_x = p_x^2/2$, it holds that

$$\langle f\rangle_{\beta,0} = \sqrt{\frac{\beta}{2\pi}}\int_{\mathbb{R}} f(p_x^2/2)\mathrm{e}^{-\beta p_x^2/2}\,\mathrm{d}p_x = \sqrt{\frac{\beta}{\pi}}\int_0^\infty f(e)\mathrm{e}^{-\beta e}\,\frac{\mathrm{d}e}{\sqrt{e}}$$

from which (6.8) follows.

Next, $\gamma^2(e_x, e_{x+1})$ is computed by means of Lemma C.1:

$$\begin{aligned} \gamma^2(e_x, e_{x+1}) &= \Pi\, j_{x,x+1}(-L_0)^{-1}j_{x,x+1} \\ &= \frac{1}{2}\Pi\, j_{x,x+1}\big[(-L_0)^{-1}\sin(q_{x+1} - q_x)p_{x+1} - (-L_0)^{-1}\sin(q_x - q_{x+1})p_x\big] \\ &= \frac{1}{2}\Pi\, j_{x,x+1}\big[\psi_{x+1,x} - \psi_{x,x+1}\big]. \end{aligned}$$

The terms in $\cos(q_x - q_{x+1})$ in $\psi_{x+1,x}$ and $\psi_{x,x+1}$ will vanish due to the projection Π, so that we are left with

$$\gamma^2(e_x, e_{x+1}) = \frac{1}{4}\Pi(p_x + p_{x+1})\sin(q_{x+1} - q_x)\frac{4\varsigma(e_{x+1}p_x + e_x p_{x+1})\sin(q_{x+1} - q_x)}{\Delta_{x,x+1}}$$

Since $\frac{1}{(2\pi)^2}\int_{[0,2\pi]^2}\sin^2(x - y)\,\mathrm{d}x\mathrm{d}y = 1/2$, and since the projection of expressions containing uneven powers of p_x or p_{x+1} vanishes, we obtain (6.9).

The current $\alpha(e_x, e_{x+1})$ can be computed in two possible ways: directly by the definition $\alpha(e_x, x_{x+1}) = \Pi G(-L_0)^{-1} j_{x,x+1}$, or by means of the expression

$$\alpha(e_x, e_{x+1}) \;=\; \mathrm{e}^{\mathcal{U}(e_x)+\mathcal{U}(e_{x+1})}\big(\partial_{e_{x+1}} - \partial_{e_x}\big)\mathrm{e}^{-(\mathcal{U}(e_x)+\mathcal{U}(e_{x+1}))}\gamma^2(e_x, e_{x+1})$$

with $\mathcal{U}(x) = \frac{1}{2}\log x$. Both computations lead to (6.10). $\square$

References

[1] C. Bernardin, Stationary non-equilibrium properties for a heat conduction model, Phys. Rev E, **78**, no. 2, (2008).

[2] C. Bernardin and F. Huveneers, Small perturbation of a disordered harmonic chain by a noise and an anharmonic potential, Prob. Th. Rel. Fields **157**, 301-331 (2013).

[3] C. Bernardin and S. Olla, Transport Properties of a Chain of Anharmonic Oscillators with Random Flip of Velocities, J. Stat.Phys. **145**, 1224-1255 (2012).

[4] T. Bodineau, B. Helffer, Correlations, Spectral Gap and Log-Sobolev inequalities for unbounded spin systems, pp. 51-66 in Differential Equations and Mathematical Physics: Proceedings of an International Conference Held at the University of Alabama in Birmingham, March 16-20, 1999, edited by R. Weikard, G. Weinstein, American Mathematical Society and International Press, 2000.

[5] F. Bonetto, J.L. Lebowitz, L. Rey-bellet, Fourier's Law: a challenge to theorist, Mathematical Physics 2000, Imperial College Press, London, 2000, pp.128-150.

[6] Oliver Butterley and Carlangelo Liverani, Robustly invariant sets in fiber contracting bundle flows. Journal of Modern Dynamics, 7, 2, 255 - 267 (2013)

[7] Oliver Butterley and Carlangelo Liverani, Smooth anosov flows: correlation spectra and stability. Journal of Modern Dynamics, 1, 2:301–322, 2007.

[8] P. Carmona, Existence and uniqueness of an invariant measure for a chain of oscillators in contact with two heat baths, Stochastic Process. Appl. **117**, no. 8, 1076-1092 (2007).

[9] A. Dhar, Heat Transport in low-dimensional systems, Advances in Physics Vol. 57, No. 5, 2008, 457537.

[10] A. Dhar, J.L.Lebowitz, V. Kannan, Heat conduction in disordered harmonic lattices with energy conserving noise, Phys. rev. E , 83, 021108, (2011),

[11] W. De Roeck and F. Huveneers, Asymptotic localization of energy in non-disordered oscillator chains, arXiv:1305.5127, 1-31 (2013).

[12] D. Dolgopiat and C. Liverani, Energy transfer in a fast-slow Hamiltonian system, Commun. Math. Phys. **308**, 201–225 (2011).

[13] J. Eckmann and M. Hairer, Non-Equilibrium Statistical Mechanics of Strongly Anharmonic Chains of Oscillators, Commun. Math. Phys. **212**, 105-164 (2000).

[14] J. Fritz, Stochastic dynamics of two-dimensional infinite particle systems, J. Stat. Phys. **20** (1979), 351–359.

[15] J. Fritz, Some remarks on nonequilibrium dynamics of infinite particle systems, J. Stat. Phys. **34** (1984), no. 3-4, 539–556.

[16] J. Fritz, Gradient dynamics of infinite point systems, Ann. Prob. **15** (1987), 478–514.
[17] P. Gaspard and T. Gilbert, Heat conduction and Fourier's law by consecutive local mixing and thermalization Physical Review Letters 101, 020601 (2008).
[18] Sebastien Gouezel and Carlangelo Liverani. Banach spaces adapted to Anosov systems. Ergodic Theory and Dynamical Systems, 26, 1:189–217, 2006.
[19] C. Kipnis and S.R.S. Varadhan, Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, Comm. Math. Phys. **104**, Number 1, 1-19 (1986).
[20] T. Komorowski, C. Landim, and S. Olla, Fluctuations in Markov processes. Time symmetry and martingale approximation. Springer (2012).
[21] S. Lepri, R. Livi, A. Politi, Thermal Conduction in classical low-dimensional lattices, Phys. Rep. **377**, 180 (2003).
[22] C. Liverani and S. Olla, Toward the Fourier law for a weakly interacting anharmonic crystal, JAMS **25**, N. 2, 555583, (2012).
[23] C. Liverani, S. Olla, M. Sasada, Macroscopic Energy diffusion after weak coupling, in preparation.
[24] L. Rey-Bellet, Open Classical System, LNM XX, Springer.
[25] H. Spohn, private communication.
[26] S.R.S. Varadhan, Nonlinear diffusion limit for a system with nearest neighbor interactions II, Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), 75-128, Pitman Res. Notes Math. Ser., 283, Longman Sci. Tech., Harlow, (1993).

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